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A new asymmetric inclusion region for minimum weight triangulation

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Abstract As a global optimization problem, planar minimum weight triangulation problem has attracted extensive research attention. In this paper, a new asymmetric graph called one-sided β-skeleton is introduced. We show that the one-sided circle-disconnected ($\sqrt{2}\beta$)-skeleton is a subgraph of a minimum weight triangulation. An algorithm for identifying subgraph of minimum weight triangulation using the one-sided ($\sqrt{2}\beta$)-skeleton is proposed and it runs in $O(n^{4/3+\epsilon} + \min\{\kappa \log n, n^2 \log n\})$ time, where κ is the number of intersected segmented between the complete graph and the greedy triangulation of the point set.

Keywords Minimum weight triangulation · Inclusion region · One-sided β-skeleton

1 Introduction

Let P be a set of points in the Euclidean plane and let n denotes its cardinality. A *triangulation* of P, denoted by $T(P)$, is defined as a maximal set of non-intersecting straight-line segments connecting points in P. The *weight* of $T(P)$, denoted by $|T(P)|$, is the sum of the lengths of all edges in $T(P)$. The *minimum weight triangulation*, or MWT in short, is defined as a triangulation of P with the minimum weight. Computing MWT for a planar point set, which is certainly a global optimization problem, is a well known problem in computational geometry and has attracts extensive research attention. Its complexity status remains unknown for over 25 years. Until very recently, the problem is shown to be NP-hard [14].

The approaches for computing MWT can be classified in two categories. The first category is to compute approximations. Existing algorithms include a constant factor approximation algorithm proposed in [12] and a quasi-polynomial time approximation scheme proposed in [15]. The second category is to identify subgraphs of the MWT, where the results can be further divided into inclusion region method and LMT-skeleton method.

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The *inclusion region* refers to a region around two points x and y, and if there exists no further points in P inside this region, the edge xy belongs to MWT(P). β-*skeleton* is a well-known inclusion region (proposed in [11]) which is defined as a region formed by two disks of diameter $\beta|xy|$ passing through both x and y. If this region is empty of further points, the edge xy is included in β-skeleton and thus belongs to MWT(P) [11]. In [7], Cheng and Xu claim that edges in 1.1768-skeleton are in MWT, which is close to $\beta = 1.154$ where a counterexample is available. Aichholzer et al. [2] shows that for convex or near-convex polygon edges β-skeleton with $\beta > 1.154$ are in MWT(P). In [17], Wang and Yang show that 1.1603 is a lower bound for β for a general point set. Yang et al. [18] gives a different inclusion region: if the union of the two disks centered at x and y with radius $|xy|$ is empty of points, then xy belongs to MWT. It is interesting to note that the above inclusion regions are all symmetric. One might expect that an asymmetric inclusion region may have the potential to produce a larger subgraph for MWT. According to the best of the author's knowledge, there is only one work on asymmetric inclusion region which is [16]. We will review their work in Sect. 1.1.

The *locally minimum triangulation* is a triangulation if every point-empty quadrilateral drawn from it is optimally triangulated. $LMT(P)$ is the intersection of all locally minimal triangulation for P [8]. One easily sees that $LMT(P)$ is a subgraph of MWT(P). LMT-skeleton is a subgraph of $LMT(P)$ and can be identified by a simple method as described in [3]. However, LMT may generate linear number of disconnected components even for uniformly distributed points [4].

Although minimum weight triangulation problem is NP-hard, it is still interesting to identify the large subgraph of it as the more edges we identified, the better chance MWT can be completed in a short time by some existing methods. For example, recent research work shows that given a point set with k inner points, MWT can be computed in $O(6^k n^5 \log n)$ time [9], and given k components of P, MWT can be completed in $O(n^{k+2})$ time [6]. Furthermore, identifying subgraph deepens our understanding on MWT, which may eventually help us design a better approximation scheme for MWT.

1.1 Previous work on asymmetric inclusion region

The main motivation for us to study the asymmetric inclusion region is to identify more edges which belong to an MWT. According to the best of the author's knowledge, Wang et al.'s work in [16] is the only result on the asymmetric inclusion region. More precisely, it proposes a sufficient condition for edge-testing rather than some fixed region like β-skeleton. Two conditions are presented in [16] and we will describe them in turn.

The first condition is that the edge xy belongs to MWT(P) if all of the following are satisfied: "(1) the interior of the triangle formed with x, y and any point above xy contains no points, i.e., *star-shaped* condition; (2) for any two points v_i, v_j above xy, $|xy| < \min\{|xv_i|, |yv_j|\}$ and no points are contained in union of two ellipses, which has v_i, v_j as foci and x or y as a boundary point, within the fan-area bounded by $v_i y$ and $v_j x$. It is the *ellipse-disconnected* condition. (3) The diameter of points below xy (inclusive) is equal to $|xy|$, where the *diameter* of a planar point set is the longest segment with both endpoints in the set" [16].

The second condition is that the edge xy belongs to MWT(P) if all of the following are satisfied: "(1) for any two points v_i above xy and v_j below xy, the diameter of points above xy is smaller than $\min\{|v_i v_j|\}$, i.e., the *thin* condition; (2) for any point v above xy, $|xy| < \min\{|xv|, |yv|\}$ and no points below xy is contained in the circle centered at v

and with vx or vy as radius within the fan-area bounded by vx and vy. (3) The diameter of points below xy (inclusive) is equal to $|xy|$" [16].

Although the work in [16] is original, it can be seen that the above conditions in [16] are not easy to satisfy in general, and thus a novel condition loosening the above requirement would have the potential to identify more edges, which motivates this work.

In this paper, a new asymmetric graph called one-sided β-skeleton, which loosens the conditions in [16], is introduced. We show that the one-sided circle-disconnected $(\sqrt{2}\beta)$-skeleton is a subgraph of a minimum weight triangulation. The new skeleton can find edges which cannot be identified using other inclusion region methods. An algorithm for identifying subgraph of minimum weight triangulation using the one-sided $(\sqrt{2}\beta)$-skeleton is proposed and it runs in $O(n^{4/3+\epsilon} + \min\{\kappa \log n, n^2 \log n\})$ time, where κ is the number of intersected segmented between the complete graph and the greedy triangulation of the point set.

The rest of the paper is organized as follows: Sect. 2 presents the definitions and the brief overview of Keil's framework [10]. Section 3 presents the major results of this paper. Section 4 presents an algorithm to identify the new inclusion region. A summary of work is given in Sect. 5.

2 Preliminaries

2.1 Definition

Recall that β-skeleton is defined as follows. An edge xy is included in β-skeleton if two disks of diameter $\beta|xy|$ passing through both x and y are empty of points. For simplicity, the one-sided β-skeleton refers to the upper part of a β-skeleton throughout the paper. We now define the concept of "circle-disconnected" as in [16]. The points above a line segment xy, denoted by V_{xy}^{+}, are called *circle-disconnected* if for any point v above xy, the following conditions hold:

1. The fan-area bounded by vx and vy below xy are empty of points.
2. For any vertex v_j below xy such that vv_j intersects the interior of the segment xy, we have

$$d(v, v_j) > \max_{v_i \in V_{xy}^{+}} d(v, v_i),$$

where $d(\cdot, \cdot)$ is the Euclidean distance function. Refer to Fig. 1 for the circle-disconnected condition. A one-sided circuit-disconnected $(\sqrt{2}\beta)$-skeleton is defined as a $(\sqrt{2}\beta)$-skeleton with one side of the β-shape being circuit-disconnected. One easily sees the following fact (as observed in [10] for the case of double-sided β-skeleton):

Observation 2.1 For any $v \in V_{xy}^{+}$ outside one-sided $(\sqrt{2}\beta)$-skeleton, $\angle xvy < \pi/3$.

In this paper, we will show that if an edge xy is in the one-sided *circle-disconnected* $(\sqrt{2}\beta)$-skeleton (or $\sqrt{2}$-skeleton in short), then the edge xy is in MWT(P). Compared to the first condition in [16], our condition does not need star-shaped condition and that the diameter for points below xy is $|xy|$, and compared to their second condition, our condition does not need thin condition and that the diameter for points below xy is $|xy|$. It is also interesting to compare the new subgraph to the (1.1768β)-skeleton as shown in Fig. 2. In Fig. 2a, since there is a point in the lower disk of the (1.1768β)-skeleton, the edge does not belong to the MWT. However, it is not hard to see that the edge is a subgraph of MWT

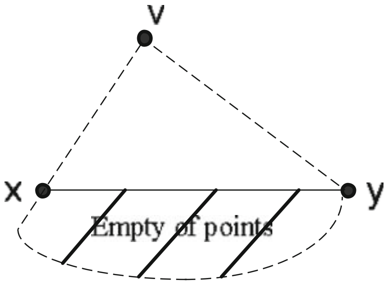


Fig. 1 Circle-disconnected condition

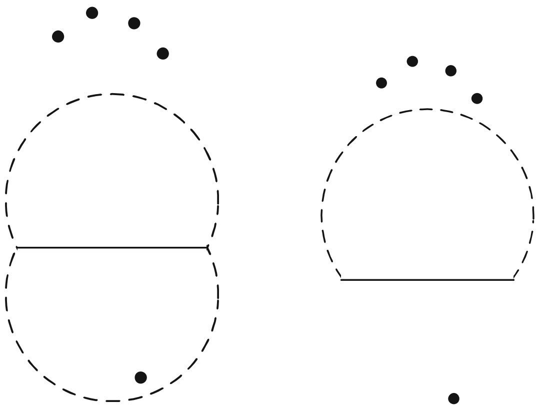


Fig. 2 Comparison of (1.1768β)-skeleton and one-sided circle-disconnected $(\sqrt{2}\beta)$-skeleton: (**a**) (1.1768β)-skeleton, (**b**) one-sided circle-disconnected $(\sqrt{2}\beta)$-skeleton

according to our condition as shown in Fig. 2b. One could also see the limitation of the new subgraph. If we move all the points to be barely outside the (1.1768β)-skeleton (very close to the boundary), they will not be outside the $(\sqrt{2}\beta)$-skeleton in either side. Subsequently, the edge does not belong to MWT by the new condition. However, it belongs to MWT since it is in the (1.1768β)-skeleton.

Our proof follows the framework proposed by Keil in [10] and then adopted by Cheng and Xu in [7]. For completeness, we include the description of the framework as follows.

2.2 Brief overview to Keil's theorem [10, 7]

Suppose to the contrary, there exists an edge xy in the one-sided circle-disconnected $\sqrt{2}$-skeleton but not in MWT(P). As such, we insert xy to MWT(P), remove all intersecting edges and re-triangulate the two resulting polygons on either side of xy. We are to show that the resulting triangulation will have strictly less weight than that of MWT(P) [10].

Assuming that there are k edges intersecting xy in MWT(P). We first sort these edges in increasing length and obtain $E' = \{e_i, 1 \le i \le k\}$ such that $|e_{i-1}| \le |e_i|, 2 \le i \le k$. For simplicity, we refer to the endpoints of edges in E' above xy as "upper" endpoints and other endpoints as "lower" endpoints. The polygonal region formed by xy and the upper endpoints of E' is incrementally re-triangulated as follows. Suppose that we have computed a sequence of triangulated polygon $P_1, P_2, \ldots, P_{i-1}$. P_i is equal to P_{i-1} if the upper endpoint v_i of e_i is inside P_{i-1}. Otherwise, P_i is the union of P_{i-1} with a newly triangulated polygon P_i'. In this case, e_i must intersect some boundary of P_{i-1} and denote the *closest* intersecting boundary edge by $v_a v_b$. $\triangle v_a v_i v_b$ contains a set of some upper endpoints, denoted by V_i, of E'. We compute a simple path through linking all vertices of the set $\{v_a, v_b, v_i\} \cup V_i$ *from left to right* and arbitrarily triangulate the polygon formed by the simple path and $v_a v_b$. The resulting polygon is P_i' [10,7].

A critical observation is that every inserted edge in P_i is shorter than e_i. We first inductively assume that every inserted edge in P_{i-1} is shorter than e_{i-1}. Clearly, each new edge in P_i (precisely, in P_i') has length at most $\max\{|v_a v_i|, |v_a v_b|, |v_i v_b|\}$. Note that $v_a v_b$ is shorter than e_{i-1} and thus e_i by induction assumption [10,7]. We only describe how to prove $|v_a v_i| < |e_i|$ and $v_i v_b$ can be similarly handled.

- If v_a lies in $\triangle x v_i y$ (note that v_i can be x or y), by Lemma 3.2 below, $v_a v_i$ is shorter than e_i.
- If v_a does not lie in $\triangle x v_i y$, then v_a must lie in a triangle $v_m v_n v_i$, where v_m and v_n are hull vertices on the convex hull of the chain from x to v_i on P_i. Therefore, $|v_a v_i|$ is at most $\max\{|v_m v_i|, |v_m v_n|, |v_n v_i|\}$. Note that in processing e_i, only v_i may become a hull vertex and this is true for $e_1, e_2, \ldots, e_{i-1}$. Since v_a and v_b are hull vertices, v_m and v_n must be added before processing e_i and they are not added as vertices in V_j, $j = 1, 2, \ldots, i-1$. That is, their corresponding edges e_m, e_n must satisfy $|e_m| \le |e_i|$ and $|e_n| \le |e_i|$. Applying Lemma 3.3 to $v_m v_i$, we have that $|v_m v_i| < \max\{|e_m|, |e_i|\} = |e_i|$. Similarly, we get $|v_n v_i|, |v_m v_n| < |e_i|$ and reach that $|v_a v_i| < |e_i|$.

See [10,7] for the further details about the framework. Note that the proof for the part below xy is symmetric to the proof just described since the conclusions in Lemma 3.2 and Lemma 3.3 are symmetric. Therefore, in this way we inductively obtain a new triangulation with weight strictly less than that of MWT(P), which leads to contradiction.

Summarizing the above, we reach the following theorem:

Theorem 2.2 *In a planar point set P, let x, y be the endpoints of an edge in the one-sided circle-disconnected $(\sqrt{2}\beta)$-skeleton, then edge xy belongs to MWT(P).*

3 Main contribution

Lemma 3.1 *For any θ, γ where $0 < \gamma < \theta < \pi$, the following holds*

$$\sin\theta - \sin(\theta - \gamma) < \frac{\sin\gamma}{\sin\theta}.$$

Proof

$$\begin{aligned}
& \sin\theta - \sin(\theta-\gamma) < \frac{\sin\gamma}{\sin\theta} && (1)\\
\Leftrightarrow\ & 2\cos\frac{2\theta-\gamma}{2}\sin\frac{\gamma}{2} < \frac{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{\sin\theta} && (2)\\
\Leftrightarrow\ & \cos\frac{2\theta-\gamma}{2} < \frac{\cos\frac{\gamma}{2}}{\sin\theta} && (3)\\
\Leftrightarrow\ & \left(\cos\theta\cos\frac{\gamma}{2} + \sin\theta\sin\frac{\gamma}{2}\right)\sin\theta < \cos\frac{\gamma}{2} && (4)\\
\Leftrightarrow\ & \sin\theta\cos\theta + \sin^2\theta\tan\frac{\gamma}{2} < 1. && (5)
\end{aligned}$$

This is true by noting that

$$
\begin{aligned}
& \sin\theta \le 1 && (6)\\
\Leftrightarrow\ & \sin\theta\cos\theta + (1-\cos\theta)\sin\theta \le 1 && (7)\\
\Leftrightarrow\ & \sin\theta\cos\theta + \left(1-\cos^2\theta\right)\frac{\sin\theta}{1+\cos\theta} \le 1 && (8)\\
\Leftrightarrow\ & \sin\theta\cos\theta + \sin^2\theta\frac{\sin\theta}{1+\cos\theta} \le 1 && (9)\\
\Leftrightarrow\ & \sin\theta\cos\theta + \sin^2\theta\tan\frac{\theta}{2} \le 1, && (10)
\end{aligned}
$$

and that $\gamma < \theta$. □

Lemma 3.2 *Let xy be an edge in the circle-disconnected $(\sqrt{2}\beta)$-skeleton of a set P of points in the plane. Let p, q be two points on the different sides of the line through xy such that pq intersects the interior of segment xy. Suppose that p is above xy. Let s be any point inside $\triangle pxy$ but outside circle-disconnected $(\sqrt{2}\beta)$-skeleton. Let r be any point inside $\triangle qxy$. Then $|px|, |py|, |qx|, |qy|, |ps|, |qr| < |pq|$.*

Proof Refer to Fig. 3. According to the circle-disconnected condition, $|px|, |py| < |pq|$. Observation 2.1 says $\angle xpy < \pi/3$, which implies $|xy| < |pq|$. ps is an edge inside $\triangle pxy$, thus $|ps| < |pq|$.

We then prove $|qx|, |qy| < |pq|$. Since q is outside the circle-disconnected region, we can compute a point x' on pq such that $|px| = |px'|$. Since $\angle xpq < \angle xpy < \pi/3$, $\angle pxx' > \pi/3$.

Fig. 3 Proof of Lemma 3.2

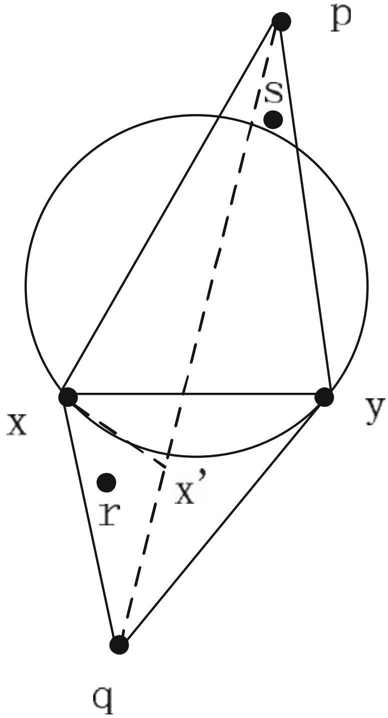
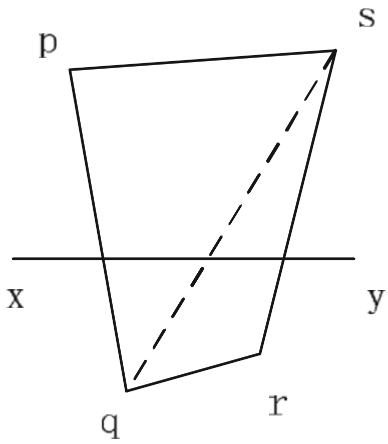


Fig. 4 pq and rs intersect below xy



It follows that $\angle pxq > \pi/3$. Therefore, $|xq| < |pq|$. Similarly, we can show $|yq| < |pq|$. Since r lies in $\triangle qxy$, $|qr| < |pq|$. □

Lemma 3.3 *Let x and y be the endpoints of an edge in the one-sided circle-disconnected $(\sqrt{2}\beta)$-skeleton of a set P of points in the plane. Let p, q, r, s be four other distinct points of P such that pq intersects the interior of xy, rs intersects the interior of xy, pq and rs does not intersect the interior of each other and p and s lie on the same side of the line through xy. Then $|qr|, |ps| < \max\{|pq|, |rs|\}$.*

Proof Without loss of generality, assume that qr is below xy and ps is above xy, and $|xy| = 1$. Note that $|ps| < |pq|, |rs|$ by the circle-disconnected condition. We first consider the situation when lines extending pq, rs are parallel or intersect below xy. Suppose to the contrary that qr is the longest edge among ps, pq, rs (refer to Fig. 4), then in $\triangle sqr$, $\angle qsr > \angle rqs$. Since $|ps| < |pq|$, $\angle pqs < \angle psq$. Therefore, $\angle pqr < \angle psr$. Similarly, we have $\angle qrs < \angle spq$. It follows that $\angle spq + \angle psr > \pi$ since $pqrs$ forms a convex quadrilateral. This contradicts the assumption that pq and rs intersect below xy or they are parallel.

The hard part is when two extended lines intersect above xy. Since it is a $(\sqrt{2}\beta)$-skeleton, we can build a square with a side being xy. Refer to Fig. 5 where $|xy| = |xa| = 1$. We first consider the case where pq, rs intersect the interior of ab. Let x' and y' be the intersection of pq and xy, and rs and xy, respectively. $|x'y'| \le 1$. Denote the angles α, β, γ as shown in Fig. 5. Without loss of generality, assume that $\alpha \ge \beta$. Clearly, $\alpha + \beta > \frac{2\pi}{3}$ by Observation 2.1.

We begin with a simple case where $r = y'$. Let p', s' denote the intersection of ab with pq and rs, respectively. Denote $|xa|$ by h and $|p's'|$ by m. We are to show $|p'q| > |qr|$ (then trivially $|qr| < |pq|$). Since

$$|qr| = \frac{|x'y'|}{\sin\gamma} \cdot \sin\alpha, \tag{11}$$

$$|p'q| = \frac{|x'y'|}{\sin\gamma} \cdot \sin(\alpha - \gamma) + \frac{h}{\sin\alpha}, \tag{12}$$

$$|x'y'| = m + \frac{h}{\tan\alpha} + \frac{h}{\tan\beta}, \tag{13}$$

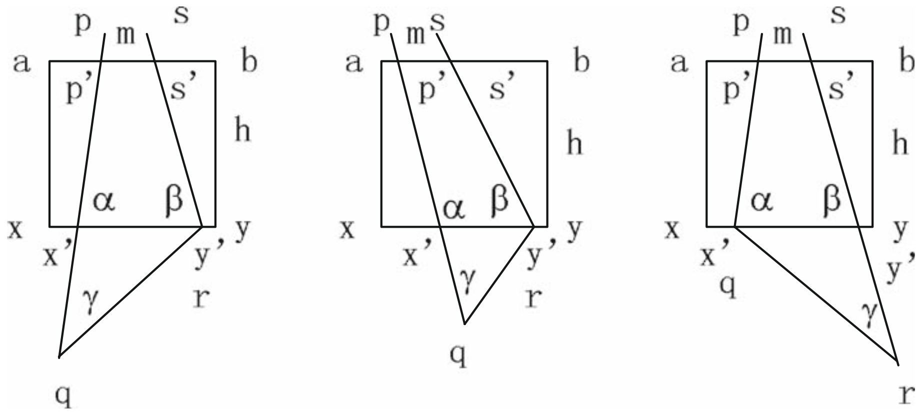


Fig. 5 pq, rs intersect above xy and intersect the interior of ab

$|p'q| > |qr|$ holds if

$$\frac{|x'y'|}{h} \cdot [\sin\alpha - \sin(\alpha-\gamma)] < \frac{\sin\gamma}{\sin\alpha}. \tag{14}$$

Note that $h = 1$ and $|x'y'| \le 1$, so $|p'q| > |qr|$ holds if

$$[\sin\alpha - \sin(\alpha-\gamma)] < \frac{\sin\gamma}{\sin\alpha}. \tag{15}$$

Note that α can be larger than $\frac{\pi}{2}$ as shown in Fig. 5, however, this will not influence the proof as $|x'y'|$ remains the same as

$$|x'y'| = m + \frac{h}{\tan\beta} - \frac{h}{|\tan\alpha|} = m + \frac{h}{\tan\beta} + \frac{h}{\tan\alpha}. \tag{16}$$

We thus only need to verify that Eq. 15 holds for $\frac{\pi}{3} \le \alpha \le \pi$ and $\beta, \gamma < \alpha$ since $\alpha+\beta \ge \frac{2\pi}{3}$, which is true according to Lemma 3.1. We also need to consider the case where $q = x'$ as shown in Fig. 5. In that case, we carry out the above process and find that the claim is valid if

$$[\sin\beta - \sin(\beta-\gamma)] < \frac{\sin\gamma}{\sin\beta}, \tag{17}$$

where $0 < \gamma < \beta \le \frac{\pi}{2}$. This is also true according to Lemma 3.1.

We now move endpoints of qr downward. The effect of this is that we introduce another "height", denoted by h', to the graph as shown in Fig. 6, where

$$|q'r| = |x'y'| + \frac{h'}{\tan\alpha} + \frac{h'}{\tan\beta}. \tag{18}$$

Equation 14 becomes

$$\frac{|q'r|}{h+h'} \cdot [\sin\alpha - \sin(\alpha-\gamma)] < \frac{\sin\gamma}{\sin\alpha}. \tag{19}$$

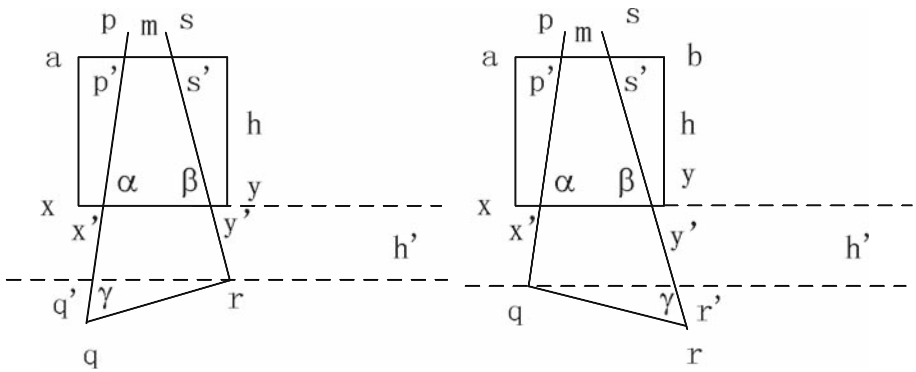


Fig. 6 Moving r

We have

$$\frac{|q'r|}{h+h'} \tag{20}$$

$$= \frac{|x'y'| + \frac{h'}{\tan\alpha} + \frac{h'}{\tan\beta}}{h+h'} \tag{21}$$

$$= \frac{m + \frac{h}{\tan\alpha} + \frac{h}{\tan\beta} + \frac{h'}{\tan\alpha} + \frac{h'}{\tan\beta}}{h+h'} \tag{22}$$

$$< \frac{|x'y'|}{h}. \tag{23}$$

Therefore, the claim still holds. We similarly prove another case as shown in Fig. 6. □

4 The algorithm

Our algorithm for identifying edges in one-sided circle-disconnected $\sqrt{2}$-skeleton is similar to the method in [16]. One easily sees that xy in one-sided circle-disconnected $\sqrt{2}$-skeleton satisfies the condition that any edge in $P \times P$ crossing xy is longer than xy, i.e., xy is a *light edge*. Since light edges form a subgraph of the greedy triangulation GT of P, we can compute it in $O(n)$ time [13] from a Delaunay triangulation. We then take each edge in $GT(P)$ as xy in Lemma 3.2 and check whether it satisfies our condition. More precisely, we have to compute all edges intersecting any edge in $GT(P)$, and in the worst case, we can have $\Theta(n^3)$ *intersecting pairs* (i.e., two segments intersect the interior of each other). However, usually the number is much smaller and we describe an algorithm for computing the intersecting pairs as follows.

Given a segment l, all segments intersected by l can be reported in $O(n^{1+\epsilon}/\sqrt{s}+\kappa)$ time with $O(s^{1+\epsilon})$ time for preprocessing where s ($n^{1+\epsilon} \le s \le n^{2+\epsilon}$) is a parameter [1] and κ is the number of total segments returned. This means that we can find segments intersected by all linear number of segments in greedy triangulation in $O(n^{4/3+\epsilon}) + \kappa$ time (including preprocessing time) by setting $s = n^{4/3}$.

Given the intersected segments, we are to verify the first condition and the second condition in the circle-disconnected condition. We begin with the second condition. For a segment

xy, we have obtained all κ_{xy} intersected segments. For each of endpoints above xy, we compute the maximum distance to the endpoints of intersected segments in $O(\kappa_{xy})$ time, and perform the test of $d(v, v_j) > \max_{v_i \in V_{xy}^+} d(v, v_i)$ for all v in $O(\kappa_{xy})$ time. Thus, verifying the second condition in circle-disconnected definition needs $O(\kappa)$ time.

Since our one-sided β-skeleton is part of a β-skeleton, we need to take the running time for computing the β-skeleton into account. β-skeleton involves computing the diameter of points. For each query, we have computed all intersected segments from the previous step and thus we know all the points. The diameter can be computed in $O(\kappa \log \kappa)$ time. Note that we have only linear number of points and thus the total running time is also bounded above by $O(n^2 \log n)$.

Both β-skeleton test and the first condition in the circle-disconnected test involve range query. We need to perform range counting for fan-areas of each segment in the greedy triangulation. A fan-area is certainly a *Tarski cell* and is defined by a constant number of constant-degree polynomials. We can therefore linearize it and answer the resulting constant-dimensional half-space range counting in $O(\log n)$ time per query by [5]. There are $O(\kappa)$ queries, and thus the total time is $O(\kappa \log n)$. We reach the following theorem:

Theorem 4.1 *Identifying subgraph for* MWT(P) *using one-sided* $(\sqrt{2}\beta)$*-skeleton can be performed in* $O(n^{4/3+\epsilon} + \min\{\kappa \log n, n^2 \log n\})$ *time, where* κ *is the total number of intersected segments and* ϵ *is any positive constant.*

5 Conclusion

This paper studies a global optimization problem, namely, the planar minimum weight triangulation problem. A new asymmetric inclusion region for identifying a subgraph of the minimum weight triangulation is proposed. An algorithm for identifying subgraph using the one-sided $(\sqrt{2}\beta)$-skeleton is proposed and it runs in $O(n^{4/3+\epsilon} + \min\{\kappa \log n, n^2 \log n\})$ time, where κ is the number of total intersected segments. Identifying subgraph deepens our understanding on MWT, which may eventually help us design a better approximation scheme for MWT.

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